

TRIANGLE CEVIAN AND SIDE RELATIONS FOR THE CONCURRENT CASE AND THE GENERAL NON-CONCURRENT CASE.

Author: Wangechi Antony Kiragu.

Email: 2205697@students.kcau.ac.ke or 3model33model3@gmail.com

Faculty: School of Technology.

Department: Department of Software Development & Information.

Institutional Affiliation: KCA University.

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I. ABSTRACT.

I began this research study by first going through some existing work on triangle geometry and I came across some interesting theorems, namely Ceva's theorem, Menelaus' theorem, Steiner-Routh's theorem and Van Aubel's theorem. By studying the above theorems and through some friends I realized that I could develop a new approach of studying and analyzing the cevian and side segments of any triangle using a set of six linear equations that I have derived in this paper.

The main contribution of this study is the proving Ceva's theorem and Menelaus' theorem, using a set of six equations derived using vectors. The equations are based on the proportions of the sides and cevians of a triangle and provide a unique and unconventional approach to solving problems in this field. One of the unique aspects of this approach is the use of vectors to derive the six equations.

This paper presents the equations together with their derivations. I have shown how the six equations can be used as the basis of proving some famous triangle theorems. In addition to proving these existing theorems, I have also proven some relatively uncommon results in triangle geometry that can be useful for further research in this area. This therefore shows that these equations have the potential to reveal even deeper concepts on triangle Geometry that may have previously been unknown in triangle geometry.

II. KEY WORDS.

Side, non-concurrent cevians, the six linear equations, Ceva's theorem, Menelaus' theorem.

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1.0 CHAPTER 1.

1.1 INTRODUCTION.

For ages, triangles have been a subject of study for many mathematicians, and indeed their properties and the relations of their components to each other are quite staggering. In this research paper, I will describe the properties of cevians and proportions inside a triangle. Specifically I will look at the various relations that exist between the sides and cevians of a triangle and consider how these can be utilized to derive some of the well-known theorems as well as some esoteric results in triangle geometry. Many studies have been done to analyze cevians and the triangle-side-segments involved by many scholars and below we will look at a few such studies.

One such study is ‘Generalizations of the theorem of Ceva and their applications’ by Florentin Smarandache from University of New Mexico, Gallup, NM 87301,USA [7].In this paper the author presents some generalizations of Ceva’s theorem when dealing with polygons. The author is also able to prove that the widely known Ceva’s theorem is only but a special case when certain parameters have certain fixed values as clearly brought out by his research study. The author Florentin Smarandache has authored many books and written several papers on geometry and other fields in Mathematics and is well known.

Another relevant study to this research is ‘Altitude (triangle)’ from Wikipedia [2] which is an article on altitudes of a triangle. This article focuses on 3 concurrent cevians that are necessarily altitudes. In this article unlike in my research study where I use 6 linear equations to study the triangle’s cevian and side proportions other theorems such as the geometric mean theorem, the in-center theorem and the circumcenter theorem, are used instead. In this article a formula that uses the three altitudes mentioned before to calculate the area of the triangle involved is also mentioned. The rather subtle relationship between the center of the nine point circle and the other centers of a triangle is shown. In this article it is also shown that the ortho-center of acute triangles is always located inside the triangles involved.

Finally, ‘A generalization of Ceva’s theorem’ by Joseph Lipman from University of Toronto, vol 67,No.2,Feb 1960 [8].The author of this paper studies the implications of Ceva’s theorem by extending this idea to involve higher dimensional surfaces and in doing so he is able to come up with elegant solutions describing the same.The author is able to successfully use Ceva’s theorem in studying hyper surfaces thus showing Ceva’s theorem can be extended to study more complicated surfaces as well as higher dimensions. While this result is not directly related to cevians or the six equations used in the present research, it does demonstrate the usefulness of Ceva’s theorem in studying various properties of triangles as well as polygons and hyper spaces.

This current investigation begins by considering the proportions of the sides and cevians of a triangle. I will define these proportions as follows $(a, 1 - a)$, $(b, 1 - b)$, $(v, 1 - v)$, $(k, 1 - k)$, $(h, 1 - h)$ and $(r, 1 - r)$ where a , b , v , k , h and r are all real numbers. I will first focus on the case where the cevians meet at a common point, which I will call M but towards the end of this paper, I will also deal with a case where the cevians don’t meet at a common point(the general case).This investigation therefore leads us to six linear equations relating these proportions.

These equations provide a unique unorthodox method of studying the relationships between the sides and cevians of a triangle especially the fact that we use the 6 equations as a unique set and as the basis of the study. Through these equations, I will prove two famous theorems namely Ceva's theorem and Menelaus' theorem. Ceva's theorem relates to the cevians of a triangle to the ratios of the sides of the triangle whereas Menelaus' theorem relates two sides of a triangle to their specific corresponding cevians. This approach provides a different view on these theorems, and offers a fresh way to conceive and apply them.

Apart from these well-known theorems, I will present some somewhat less famous results related to triangle geometry which can be derived from the generalized version of the six equations with aid of Menelaus' theorem. These results provide new insights into the triangle properties and give a depth understanding of the relationships in between the sides and cevians of a triangle.

It is important to mention that I used vectors to derive the six equations that serve as the basis of this study. Developing the concept in this manner is unique in the simplicity and clarity it offers, thus giving a novel and reliable new approach useful in understanding the relationships between the sides and cevians of a triangle.

2.0 CHAPTER 2.

2.1 METHODOLOGY.

2.1.1 THE SIX VECTOR BASED LINEAR EQUATIONS.

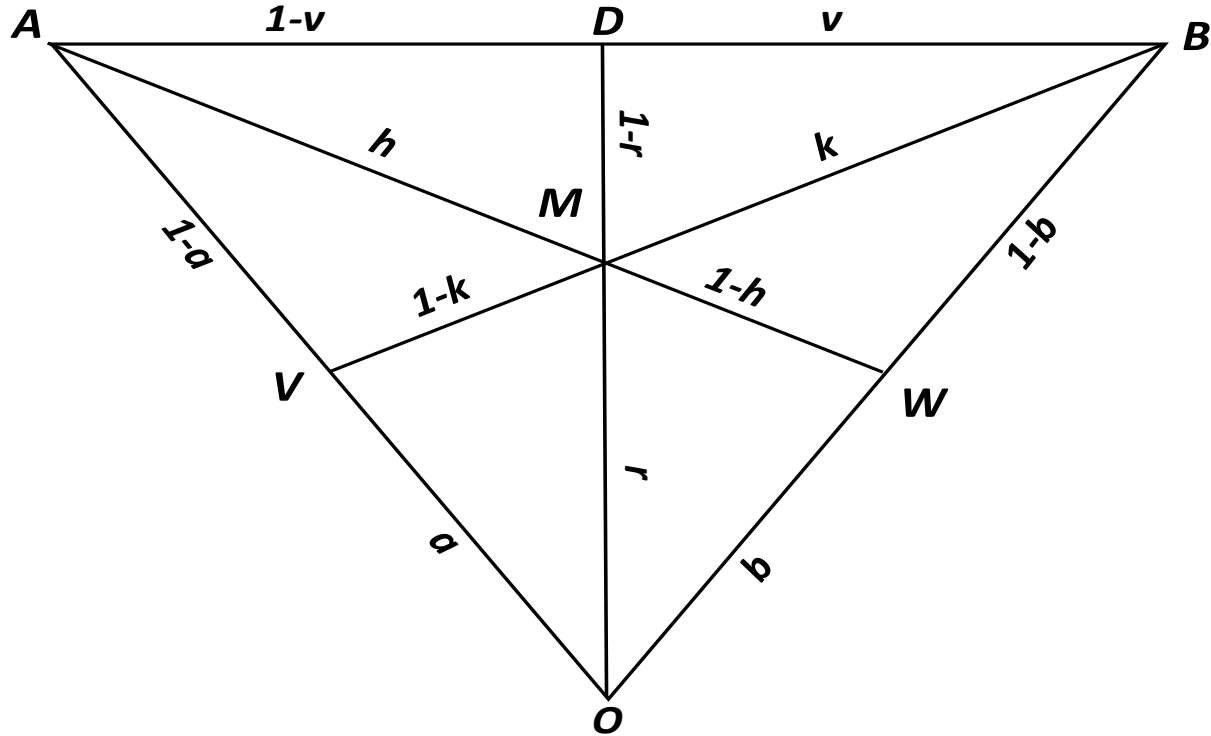


Figure 1: A triangle ABO where the proportions of the side AO are $(a, (1-a))$ and those of the cevian that meets AO at V and originates from vertex B are $(k, (1-k))$, proportions of the side BO are $(b, (1-b))$ and those of the cevian which meets BO at W and originates from vertex A are $(h, (1-h))$, proportions of the AB are $(v, (1-v))$ and those of the cevian that meets line AB at D and originates from vertex O are $(r, (1-r))$. All the three Cevians meet at M.

Assume that the proportions of the side AO are $(a, (1-a))$ and those of the cevian that meets AO at V and originates from vertex B are $(k, (1-k))$, proportions of the side BO are $(b, (1-b))$ and those of the cevian which meets BO at W and originates from vertex A are $(h, (1-h))$, proportions of the AB are $(v, (1-v))$ and those of the cevian that meets line AB at D and originates from vertex O are $(r, (1-r))$. Given that all the three cevians meet at M, the 6 equations below are coherent for any triangle (A,B,O).

$$1 - h = ak$$

$$1 - h = vr$$

$$1 - k = bh$$

$$1 - k = (1 - v)r$$

$$1 - r = (1 - a)k$$

$$1 - r = (1 - b)h$$

2.1.2 DERIVATION OF THE SIX VECTOR BASED LINEAR EQUATIONS.

Let (\mathbf{a}) and (\mathbf{b}) be 2-D vectors such that:

$$\mathbf{OA} = \mathbf{a} \text{ and } \mathbf{OB} = \mathbf{b}$$

From the above diagram the it is then obvious that

$$\mathbf{AB} = \mathbf{b} - \mathbf{a} \tag{0}$$

In order to come up with 2 vector equations we can then use the vector \mathbf{OM} as our resultant vector and express it in terms of other vectors as shown by the two equations below:

$$\mathbf{OM} = \mathbf{OA} + \mathbf{AM} \tag{1}$$

$$\mathbf{OM} = \mathbf{OB} + \mathbf{BM} \tag{2}$$

The next step is simplifying the two equations above so that the vector \mathbf{OM} can be expressed in terms of vectors (\mathbf{a}) and (\mathbf{b}) in two different ways. We shall proceed as shown below:

Equation(1).

$$\mathbf{OM} = \mathbf{OA} + \mathbf{AM} \tag{1}$$

$$\mathbf{AM} = h(\mathbf{AW}) \tag{3}$$

$$\mathbf{AW} = \mathbf{AB} + (1 - b)\mathbf{BO}$$

$$\text{but } \mathbf{BO} = -\mathbf{OB} = -\mathbf{b}$$

$$\text{hence } \mathbf{AW} = \mathbf{b} - \mathbf{a} + (1 - b) * -\mathbf{b}$$

$$\mathbf{AW} = b * \mathbf{b} - \mathbf{a}$$

Equation (3) can thus be expressed as shown below

$$\mathbf{AM} = bh * \mathbf{b} - h * \mathbf{a}$$

Equation (1) then takes the form shown below

$$\mathbf{OM} = (1 - h) * \mathbf{a} + bh * \mathbf{b} \tag{1}$$

Equation(2).

$$\mathbf{OM} = \mathbf{OB} + \mathbf{BM} \quad (2)$$

$$\mathbf{BM} = k(\mathbf{BV})$$

$$\mathbf{BV} = \mathbf{BA} + \mathbf{AV}$$

$$\mathbf{BA} = -\mathbf{AB} = \mathbf{a} - \mathbf{b} \dots (\text{from equation (0)})$$

$$\mathbf{AV} = (1 - a)\mathbf{AO} = a * \mathbf{a} - \mathbf{a}$$

$$\mathbf{BV} = \mathbf{BA} + \mathbf{AV} = a * \mathbf{a} - \mathbf{b}$$

$$\mathbf{BM} = k(\mathbf{BV}) = ka * \mathbf{a} - k * \mathbf{b}$$

Since we have defined the vectors \mathbf{OB} and \mathbf{BM} we can express the equation 2 as shown below

$$\mathbf{OM} = (1 - k)\mathbf{b} + ak * \mathbf{a} \quad (2)$$

Since $(\text{Equation}(1) = \text{Equation}(2))$ we can deduce the following:

$$ak + h = 1$$

$$bh + k = 1$$

We need to express the resultant vectors (\mathbf{BM}) and (\mathbf{AM}) in terms of the vectors (\mathbf{a}) and (\mathbf{b}) and come up with 4 more equations increasing the number of equations to 6.

The equations will thus be:

$$1 - h = ak$$

$$1 - h = vr$$

$$1 - k = bh$$

$$1 - k = (1 - v)r$$

$$1 - r = (1 - a)k$$

$$1 - r = (1 - b)h$$

Remark.

As for the general case where the three cevians are not concurrent the above equations will not hold, instead a different set of 6 equations will have to be used. Towards the end of this paper I will write down the equations to be used for such a case and I will use them to derive a few lesser known results in triangle geometry.

3.0 CHAPTER 3.

3.1 RESULTS.

In this section I'm going to demonstrate the significance of the 6 equations. I'm going to do this by using the 6 as the basis of proving some existing mathematical theorems .The approach I'm going to use is first proving the theorems using other well-known methods then there after I'm going to prove the theorems using the 6 equations.

The commonly known mathematical theorems we are going to derive are the Ceva's theorem and the Menelaus' theorem .The other equations I will prove will follow after.

3.1.1 MENELAUS' THEOREM.

Consider a triangle (A,B,O) such that the proportions of the side AO are ((a,(1-a)) and those of the cevian that meets AO at V and originates from vertex B are (k,(1-k)) ,proportions of the side BO are((b,(1-b)) and those of the cevian which meets BO at W and originates from vertex A are (h,(1-h)),proportions of the AB are((v,(1-v)) and those of the cevian that meets line AB at D and originates from vertex O are(r,(1-r)).All the three Cevians meet at M.

A. Proof using a traditional method.

Menelaus' Theorem in summary is about collinear points with reference to a triangle. For our case the theorem can be proven with reference to the points (V, M, B) which are points in our triangle that are indeed collinear.

The theorem then shows that if that's the case then the following equation must always be true:

$$\frac{AV}{OV} \times \frac{BO}{BW} \times \frac{MW}{AM} = 1$$

In order to prove this equation, we will begin by assuming that it is indeed true. So our task will then to prove the truth of the equation using both methods.

Draw a line WQ which should be parallel to the line OA and should join the line MB at Q as shown.

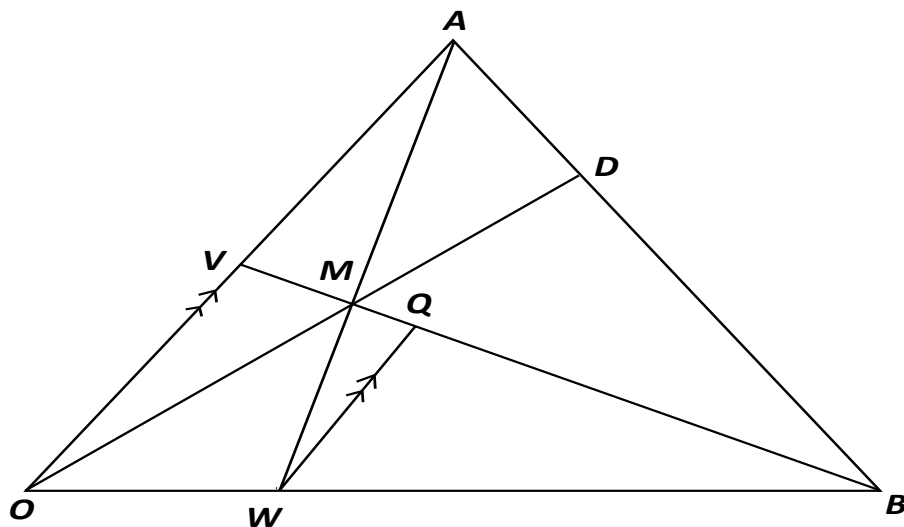


Figure 2: A triangle ABO with 3 concurrent cevians AD, BV& AW with side OA being parallel to line WQ where Q is a point inside the triangle on the cevian BV.

After doing this we are able to make the following observations:

The triangles (AVM) and (MWQ) are similar triangles, thus

$$\frac{AV}{WQ} = \frac{AM}{WM} \quad (4)$$

We can also observe that triangles (OBV) and (WBQ) are similar thus

$$\frac{OV}{BO} = \frac{WQ}{BW} \quad (5)$$

From equation (4) we can clearly see that

$$WQ = AV \times \frac{WM}{AM}$$

From equation (5) we also see that

$$WQ = OV \times \frac{BW}{BO}$$

Equating the two values we see that

$$OV \times \frac{BW}{BO} = AV \times \frac{WM}{AM}$$

Which when we rearrange takes the form

$$\frac{OV}{AV} \times \frac{AM}{MW} \times \frac{BW}{BO} = 1$$

This is Menelaus' theorem in equation form as we saw earlier.

B. Proof using the 6 linear equations.

The Menelaus' theorem stated using the ratios we defined in our problem takes the form.

$$\left(\frac{1-a}{a}\right) \times \left(\frac{1}{1-b}\right) \times \left(\frac{1-h}{h}\right) = 1 \quad (\text{p})$$

To begin our proof we first have to list the 6 equations we are going to use

$$1 - h = ak \quad (\text{a})$$

$$1 - h = vr \quad (\text{b})$$

$$1 - k = bh \quad (\text{c})$$

$$1 - k = (1 - v)r \quad (\text{d})$$

$$1 - r = (1 - a)k \quad (\text{e})$$

$$1 - r = (1 - b)h \quad (\text{f})$$

We then have to express some values in our equation (p) in a more suitable form as follows .

$$1 - a = \frac{1-r}{k} \quad (\text{e})$$

$$a = \frac{1-h}{k} \quad (\text{a})$$

$$1 - b = \frac{1-r}{h} \quad (\text{f})$$

After doing this we can then substitute these values in our equation (p) as follows.

$$\left(\frac{1-a}{a}\right) \times \left(\frac{1}{1-b}\right) \times \left(\frac{1-h}{h}\right) = \left(\frac{\frac{1-r}{k}}{\frac{1-h}{k}}\right) \times \left(\frac{h}{1-r}\right) \times \left(\frac{1-h}{h}\right) = 1$$

On dividing we are left with 1 which proves Menelaus' theorem true.

This is therefore a proof of Menelaus' theorem using the 6 equations.

3.1.2 CEVA'S THEOREM.

Consider a triangle (A,B,O) such that the proportions of the side AO are $(a,(1-a))$ and those of the cevian that meets AO at V and originates from vertex B are $(k,(1-k))$, proportions of the side BO are $(b,(1-b))$ and those of the cevian which meets BO at W and originates from vertex A are $(h,(1-h))$, proportions of the AB are $(v,(1-v))$ and those of the cevian that meets line AB at D and originates from vertex O are $(r,(1-r))$. All the three Cevians meet at M .

A. Proof using a traditional method.

Ceva's theorem is such that for the three Cevians to be concurrent then the equation below must hold for all triangles.

$$\left(\frac{AV}{VO}\right) \times \left(\frac{OW}{WB}\right) \times \left(\frac{BD}{DA}\right) = 1$$

For this proof as well we are going to start with the assumption the equation mentioned above is true. In this respect our task is then to prove the truth of the equation using the two methods involved.

For this proof we shall draw 2 lines that are both parallel to the line AW , one originating from O to a point R that is not in the triangle and another line originating from B to a point U . We shall then extend the line OD to meet the line BU at U and we shall also extend the line BV to meet line OR at R . This is illustrated in the diagram below.

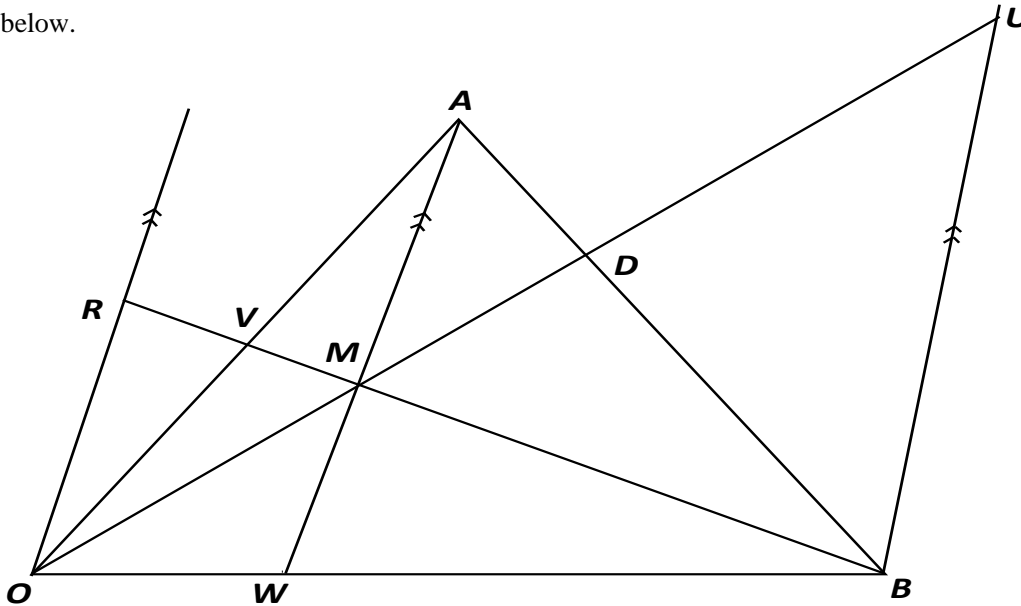


Figure 3: A triangle ABO with 3 concurrent cevians AD, BV & AW with cevian AW being parallel to lines OR and BU where U and R are points outside the triangle.

From the diagram above we can then observe the following:

Triangles (AVM) and (RVO) are similar thus

$$\frac{AV}{VO} = \frac{AM}{RO} \tag{6}$$

Triangles (OWM) and (OUB) are similar thus

$$\frac{OW}{OB} = \frac{MW}{UB} \quad (7)$$

Triangles (BWM) and (BOR) are similar thus

$$\frac{OB}{BW} = \frac{RO}{MW} \quad (8)$$

Triangles (BUD) and (ADM) are similar thus

$$\frac{BD}{DA} = \frac{UB}{AM} \quad (9)$$

Multiplying these four equations we get the following

$$\frac{AV \times OW \times OB \times BD}{VO \times OB \times BW \times DA} = \frac{AM \times MW \times RO \times UB}{RO \times UB \times MW \times AM}$$

Simplifying the above equation we get

$$\frac{AV}{VO} \times \frac{OW}{BW} \times \frac{BD}{DA} = 1$$

Which is indeed the Ceva's theorem.

B. Proof using the 6 linear equations.

To start this proof we start by listing the 6 equations we are going to use.

They are:

$$1 - h = ak \quad (a)$$

$$1 - h = vr \quad (b)$$

$$1 - k = bh \quad (c)$$

$$1 - k = (1 - v)r \quad (d)$$

$$1 - r = (1 - a)k \quad (e)$$

$$1 - r = (1 - b)h \quad (f)$$

Ceva's theorem can be expressed using the ratios defined in our question as follows

$$\left(\frac{a}{1-a}\right) \times \left(\frac{1-v}{v}\right) \times \left(\frac{1-b}{b}\right) = 1 \quad (m)$$

Our task then is to prove this equation using our 6 equations.

First we need to express the relevant values in the above equation (m) using the 6 linear equations I derived as follows

$$a = \frac{1-h}{k} \tag{a}$$

$$1 - a = \frac{1-r}{k} \tag{e}$$

$$1 - v = \frac{1-k}{r} \tag{d}$$

$$v = \frac{1-h}{r} \tag{b}$$

$$1 - b = \frac{1-r}{h} \tag{f}$$

$$b = \frac{1-k}{h} \tag{c}$$

We then substitute these values in our equation (m). After doing this we get

$$\left(\frac{a}{1-a}\right) \times \left(\frac{1-v}{v}\right) \times \left(\frac{1-b}{b}\right) = \left(\frac{\frac{1-h}{k}}{\frac{1-r}{k}}\right) \times \left(\frac{\frac{1-k}{r}}{\frac{1-h}{r}}\right) \times \left(\frac{\frac{1-r}{h}}{\frac{1-k}{h}}\right) = 1$$

On cancellation of common terms we are left with (1) as our answer. This therefore confirms that Ceva's theorem is true.

This therefore is the proof of Ceva's theorem using the 6 linear equations.

3.2 A WORKED EXAMPLE.

I would like to emphasize that based on my work the significance of the 6 equations we derived earlier is not solving numerical problems since theorems such as the angle bisector theorem, Ceva's theorem and Menelaus' theorem can do the same with relative ease. The real significance of the 6 equations as brought out by my work is in deriving and proving already existing and less known results/solutions as we saw earlier.

In order to show the accuracy of the 6 equations it's only sensible to solve a numerical problem to prove the same. In this regard below is such a problem. In solving this problem I have used Menelaus' theorem to first solve the problem and below it I have presented the solution to the problem using the 6 linear equations we derived earlier. For this example I will only consider the case where the cevians are concurrent.

The diagram below shows a triangle (ABO). In the diagram we can observe the proportions of the three sides of the triangle which are a direct result of three cevians which intersect the three sides. The three Cevians involved are concurrent and M is the point of concurrence of the three cevians. The proportions of the three sides are side AO → (AV:VO = 3:2), side AB → (AD:DB = 7:9) and side OB → (OW:WB = x:4) where (x) is an unknown value.

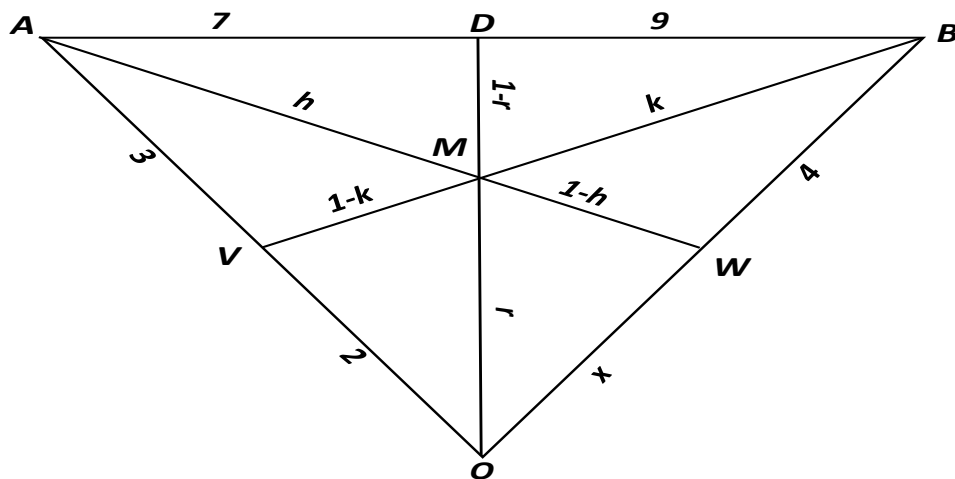


Figure 4: A triangle ABO where the proportions of the side AO are $(\frac{2}{5}, \frac{3}{5})$ and those of the cevian that meets AO at V and originates from vertex B are $(k, (1-k))$, proportions of the side BO are $(\frac{x}{4+x}, \frac{4}{4+x})$ and those of the cevian which meets BO at W and originates from vertex A are $(h, (1-h))$, proportions of the AB are $(\frac{7}{16}, \frac{9}{16})$ and those of the cevian that meets line AB at D and originates from vertex O are $(r, (1-r))$. All the three Cevians meet at M.

Questions:

1. Find (x)
2. Use the Menelaus' theorem to find the ratios of all the cevians
3. Use the 6 equations we derived earlier to find the ratios of all the cevians

Solution:

1. Find x

From Ceva's theorem we know that

$$\left(\frac{OV}{VA}\right) \times \left(\frac{AD}{DB}\right) \times \left(\frac{BW}{WO}\right) = 1$$

Thus

$$\left(\frac{2}{3}\right) \times \left(\frac{7}{9}\right) \times \left(\frac{4}{x}\right) = 1 \Rightarrow x = \left(\frac{2}{3}\right) \times \left(\frac{7}{9}\right) \times 4 = \frac{56}{27}$$

2. Use Menelaus' theorem to find the ratios of all the cevians involved

Cevian AW

From Menelaus' theorem we know that

$$\frac{AV}{VO} \times \frac{OB}{WB} \times \frac{WM}{AM} = 1$$

Therefore

$$\frac{3}{2} \times \frac{41}{27} \times \frac{WM}{AM} = 1 \Rightarrow \frac{WM}{AM} = \frac{18}{41}$$

thus the ratio of the cevian AW is ($WM:AM = 18:41$)

Cevian OD

From Menelaus' theorem we know that

$$\frac{OW}{WB} \times \frac{BA}{DA} \times \frac{DM}{MO} = 1$$

Therefore

$$\frac{14}{27} \times \frac{16}{7} \times \frac{DM}{MO} = 1 \Rightarrow \frac{DM}{MO} = \frac{27}{32}$$

Thus the ratio of the cevian OD is ($DM:MO = 27:32$)

Cevian BV

From Menelaus' theorem we know that

$$\frac{BW}{WO} \times \frac{OA}{VA} \times \frac{VM}{MB} = 1$$

Therefore

$$\frac{27}{14} \times \frac{5}{3} \times \frac{VM}{MB} = 1 \Rightarrow \frac{VM}{MB} = \frac{14}{45}$$

Thus the ratio of Cevian BV is ($VM:MB = 14:45$)

We have therefore found all the ratios for all the three cevians using Menelaus' theorem.

3. Use the 6 equations we derived to find the ratios of all the cevians involved.

In order to solve this question using the equations we can relate the triangle (ABO) to the triangle described below

Assume that the proportions of the side AO are $(a, (1-a))$ and those of the cevian that meets AO at V and originates from vertex B are $(k, (1-k))$, proportions of the side BO are $(b, (1-b))$ and those of the cevian which meets BO at W and originates from vertex A are $(h, (1-h))$, proportions of the AB are $(v, (1-v))$ and those of the cevian that meets line AB at D and originates from vertex O are $(r, (1-r))$. All the three cevians meet at M.

The equations are therefore

$$1 - h = ak \tag{a}$$

$$1 - h = vr \tag{b}$$

$$1 - k = bh \tag{c}$$

$$1 - k = (1 - v)r \tag{d}$$

$$1 - r = (1 - a)k \tag{e}$$

$$1 - r = (1 - b)h \tag{f}$$

We then substitute the known values in the equations as follows

$$1 - h = \left(\frac{2}{5}\right) \times k \tag{g}$$

$$1 - h = \left(\frac{9}{16}\right) \times r \tag{h}$$

$$1 - k = \left(\frac{14}{41}\right) \times h \tag{i}$$

$$1 - k = \left(\frac{7}{16}\right) \times r \quad (j)$$

$$1 - r = \left(\frac{3}{5}\right) \times k \quad (k)$$

$$1 - r = \left(\frac{21}{41}\right) \times h \quad (l)$$

We can then proceed to solve the linear equations we have formed

For this case we are only interested with equations (j) and (k)

$$\left(\frac{7}{16}\right) \times r + k = 1 \quad (j)$$

$$r + \left(\frac{3}{5}\right) \times k = 1 \quad (k)$$

If we multiply equation (k) by $\left(\frac{7}{16}\right)$ we get the equation below

$$\left(\frac{7}{16}\right) \times r + \left(\frac{21}{80}\right) \times k = \left(\frac{7}{16}\right)$$

Subtracting this equation from the equation (j) we get

$$\left(\frac{7}{16}\right) \times r + k - \left(\left(\frac{7}{16}\right) \times r + \left(\frac{21}{80}\right) \times k\right) = 1 - \left(\frac{7}{16}\right)$$

This gives

$$k = \left(\frac{45}{59}\right) \text{ and } r = \left(\frac{32}{59}\right)$$

We also get the following

$$1 - k = \left(\frac{14}{59}\right) \text{ and } 1 - r = \left(\frac{27}{59}\right)$$

To find h we proceed as shown below

From equation (g) we know that

$$(1 - h) = \left(\frac{2}{5}\right) \times \left(\frac{45}{59}\right) = \left(\frac{18}{59}\right)$$

$$\text{Thus } 1 - h = \left(\frac{18}{59}\right) \text{ and } h = \left(\frac{41}{59}\right)$$

Since we now know the values of all the variables we can proceed to find the ratios of the cevians as follows

$$\left(\frac{BM}{MV} = \left(\frac{45}{14} \right) \right)$$

Which gives the ratio of the cevian (BV) as

$$(BM: MV = 45: 14)$$

$$\left(\frac{AM}{MW} = \left(\frac{41}{18} \right) \right)$$

Which gives the ratio of the cevian (AW) as

$$(AM: MW = 41: 18)$$

$$\left(\frac{OM}{MD} = \left(\frac{32}{27} \right) \right)$$

Which gives the ratio of the cevian (OD) as

$$(OM: MD = 32: 27)$$

We see that using the 2 methods we arrive at the same answers thus this verifies the 6 linear equations as accurate and reliable.

4.0 CHAPTER 4.

4.1 THE GENERALIZED VERSION OF THE 6 LINEAR EQUATIONS PROPOSED BY THIS STUDY AND A FEW NEW INSIGHTS THAT THEY OFFER IN TRIANGLE GEOMETRY.

I will now rewrite the 6 linear equations so that they can describe the scenario when the three cevians in a triangle are not concurrent. I will do this so that I am able to derive more complex relationships involving non-concurrent cevians. At the end of chapter 4 I will write down the final equations of this research that will describe the relationships between the ratios of the sides of a triangle and the cevians (whether concurrent or not). This final equations will serve as the summary of my research work. It should be noted that for that under this section I will not consider the directions of the parameters used in the equations involved as done in vectors. I will only deal with the absolute values of the parameters involved for simplicity. I will not prove or derive the generalized version of the 6 linear equations as one only needs to use vectors as I did in the beginning of this paper thus I will not repeat the same as they can easily be verified and no additional theory is required.

The reader should note that the generalized version of the 6 linear equations describe figures 5, 6 and 7. I have drawn this figures separately so that the reader can be able to clearly distinguish the different ratios involved without much difficulty.

4.1.1 THE GENERALIZED VERSION OF THE 6 LINEAR EQUATIONS THAT DESCRIBE THE NON-CONCURRENT CEVIANS IN ANY TRIANGLE.

The reader should note that the three diagrams below (figures 5, 6 and 7) are essentially the same triangle and my reason for drawing them as three different triangles is for clarity to the reader and for visualization purposes only.

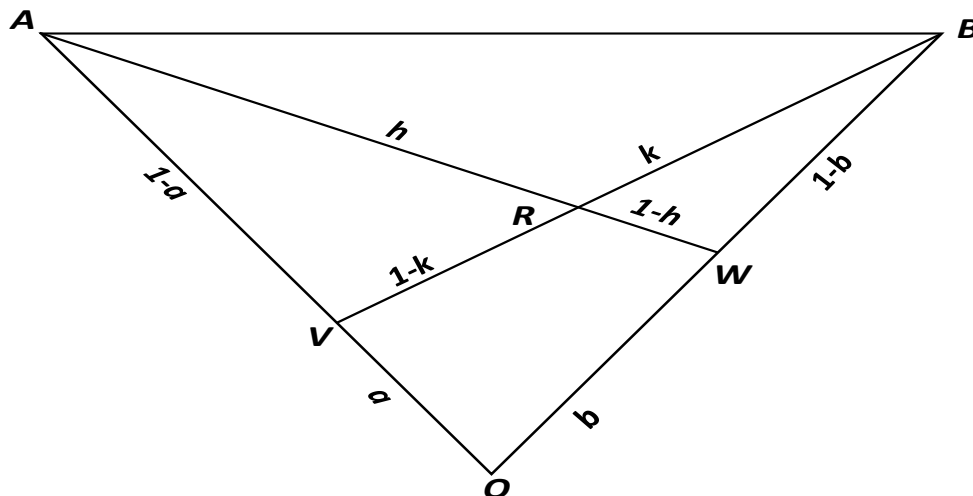


Figure 5: A triangle ABO where the proportions of the side AO are $(a, (1-a))$ and those of the cevian that meets AO at V and originates from vertex B are $(k, (1-k))$, proportions of the side BO are $(b, (1-b))$ and those of the cevian which meets BO at W and originates from vertex A are $(h, (1-h))$. The two Cevians meet at R.

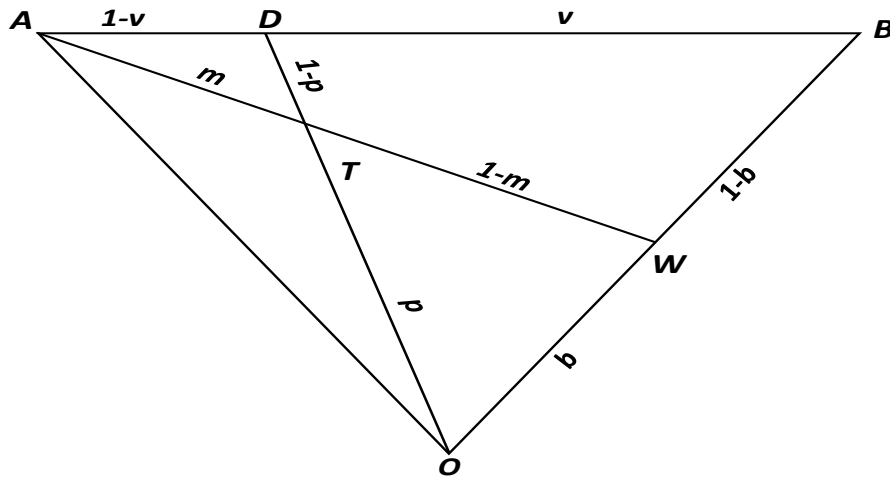


Figure 6: A triangle ABO where the proportions of the side BO are $((b, (1-b)))$ and those of the cevian which meets BO at W and originates from vertex A are $(m, (1-m))$, proportions of the AB are $((v, (1-v)))$ and those of the cevian that meets line AB at D and originates from vertex O are $(p, (1-p))$. The two Cevians meet at T.

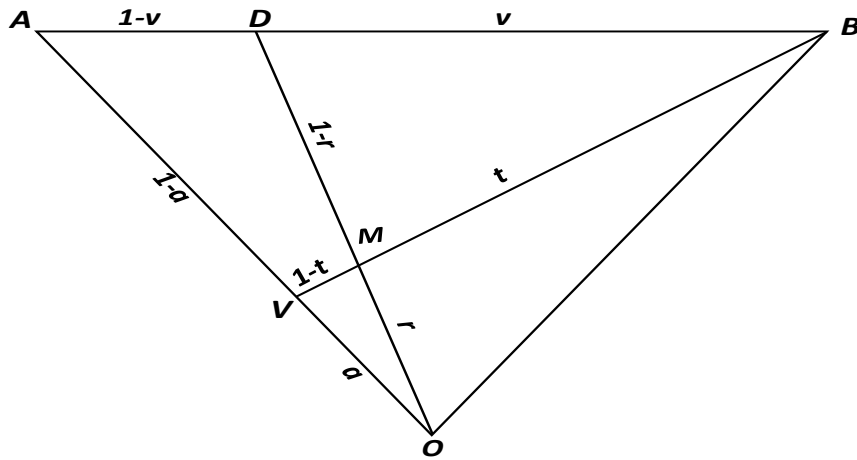


Figure 7: A triangle ABO where the proportions of the side AO are $((a, (1-a)))$ and those of the cevian that meets AO at V and originates from vertex B are $(t, (1-t))$, proportions of the AB are $((v, (1-v)))$ and those of the cevian that meets line AB at D and originates from vertex O are $(r, (1-r))$. The two cevians meet at M.

From figures 5, 6 and 7 the following 6 linear equations are true and this can very easily be verified using vectors similar to what I did with the initial 6 linear equations that were dealing with 3 concurrent cevians.

$$1 - h = ak$$

$$1 - k = bh$$

$$1 - m = vp$$

$$1 - p = m * (1 - b)$$

$$1 - t = r * (1 - v)$$

$$1 - r = t * (1 - a)$$

The 6 linear equations above indeed describe all triangles with 3 non-concurrent cevians. In the section below I will use these 6 equations together with Menelaus' theorem to prove certain relations concerning non-concurrent cevians.

4.1.2 PROVING THE MORE COMPLEX RELATIONSHIPS INVOLVING NON-CONCURRENT CEVIANS.

The first way of expressing Menelaus' theorem for each triangle.

From figure 5 Menelaus' theorem can be expressed as

$$\left(\frac{1-b}{b}\right) \times \left(\frac{1}{1-a}\right) \times \left(\frac{1-k}{k}\right) = 1$$

From figure 6 Menelaus' theorem can be expressed as

$$\left(\frac{1}{b}\right) \times \left(\frac{1-m}{m}\right) \times \left(\frac{1-v}{v}\right) = 1$$

From figure 7 Menelaus' theorem can be expressed as

$$\left(\frac{1}{v}\right) \times \left(\frac{1-r}{r}\right) \times \left(\frac{a}{1-a}\right) = 1$$

The next step is to multiply all the three equations representing Menelaus' theorem together.

After multiplying them together we get

$$\left(\frac{a}{(1-a)^2}\right) \times \left(\frac{1-b}{b^2}\right) \times \left(\frac{1-v}{v^2}\right) \times \left(\frac{1-m}{m}\right) \times \left(\frac{1-r}{r}\right) \times \left(\frac{1-k}{k}\right) = 1 \quad (10)$$

The second way of expressing Menelaus' theorem for each triangle.

From figure 5 Menelaus' theorem can also be expressed as

$$\left(\frac{1-a}{a}\right) \times \left(\frac{1}{1-b}\right) \times \left(\frac{1-h}{h}\right) = 1$$

From figure 6 Menelaus' theorem can be expressed as

$$\left(\frac{b}{1-b}\right) \times \left(\frac{1}{1-v}\right) \times \left(\frac{1-p}{p}\right) = 1$$

From figure 7 Menelaus' theorem can be expressed as

$$\left(\frac{v}{1-v}\right) \times \left(\frac{1}{a}\right) \times \left(\frac{1-t}{t}\right) = 1$$

We need then to multiply these three equations together just as we did in the first case.

On doing this we get

$$\left(\frac{1-a}{a^2}\right) \times \left(\frac{b}{(1-b)^2}\right) \times \left(\frac{v}{(1-v)^2}\right) \times \left(\frac{1-h}{h}\right) \times \left(\frac{1-t}{t}\right) \times \left(\frac{1-p}{p}\right) = 1 \quad (11)$$

The final part is dividing equations 10 and 11 together as shown below.

$$\frac{\left(\frac{a}{(1-a)^2}\right) \times \left(\frac{1-b}{b^2}\right) \times \left(\frac{1-v}{v^2}\right) \times \left(\frac{1-m}{m}\right) \times \left(\frac{1-r}{r}\right) \times \left(\frac{1-k}{k}\right)}{\left(\frac{1-a}{a^2}\right) \times \left(\frac{b}{(1-b)^2}\right) \times \left(\frac{v}{(1-v)^2}\right) \times \left(\frac{1-h}{h}\right) \times \left(\frac{1-t}{t}\right) \times \left(\frac{1-p}{p}\right)} = 1$$

When this is done we get

$$\left(\frac{(1-m) \times (1-r) \times (1-k)}{m \times r \times k}\right) \times \left(\frac{h \times t \times p}{(1-h) \times (1-t) \times (1-p)}\right) \times \left(\frac{a \times (1-b) \times (1-v)}{(1-a) \times b \times v}\right)^3 = 1 \quad (12)$$

The next step is to divide the 6 generalized linear equations in pairs as shown below.

$$\frac{1-h}{1-k} = \frac{ak}{bh} \dots \dots \text{(from figure 5)}$$

$$\frac{1-m}{1-p} = \frac{vp}{m(1-b)} \dots \dots \text{(from figure 6)}$$

$$\frac{1-t}{1-r} = \frac{r(1-v)}{t(1-a)} \dots \dots \text{(from figure 7)}$$

We then multiply the three equations above as shown below

$$\frac{1-h}{1-k} * \frac{1-p}{1-m} * \frac{1-t}{1-r} = \frac{a}{1-a} * \frac{1-b}{b} * \frac{1-v}{v} * \frac{k}{h} * \frac{m}{p} * \frac{r}{t}$$

Which simplifies to

$$\frac{a}{1-a} * \frac{1-b}{b} * \frac{1-v}{v} = \frac{h(1-h)}{k(1-k)} * \frac{p(1-p)}{m(1-m)} * \frac{t(1-t)}{r(1-r)} \quad (13)$$

With equations (12) and (13) above we can then proceed to derive our uncommon solutions.

4.1.3 SOLUTION (1).

We need to cube the equation 12 to get

$$\left(\frac{h(1-h)}{k(1-k)} \times \frac{p(1-p)}{m(1-m)} \times \frac{t(1-t)}{r(1-r)}\right)^3 = \left(\frac{a}{1-a} \times \frac{1-b}{b} \times \frac{1-v}{v}\right)^3$$

We also need to cube equation 13 to get

$$\left(\frac{a}{1-a} \times \frac{1-b}{b} \times \frac{1-v}{v}\right)^9 = \left(\frac{m}{1-m} \times \frac{r}{1-r} \times \frac{k}{1-k} \times \frac{1-h}{h} \times \frac{1-t}{t} \times \frac{1-p}{p}\right)^3$$

The next step is then to multiply two equations as shown below

$$(\text{equation 12})^3 \times \frac{1}{(\text{equation 13})^3}$$

On doing this we get

$$\left(\frac{a}{1-a} \times \frac{1-b}{b} \times \frac{1-v}{v}\right)^9 \times \left(\frac{1-a}{a} \times \frac{b}{1-b} \times \frac{v}{1-v}\right)^3 = \left(\frac{k}{h} \times \frac{m}{p} \times \frac{r}{t}\right)^6$$

Which can be simplified to

$$\frac{a}{1-a} \times \frac{1-b}{b} \times \frac{1-v}{v} = \frac{k}{h} \times \frac{m}{p} \times \frac{r}{t}$$

This therefore is our first uncommon solution in this section. The above equation can also be written as shown below

$$\frac{OV}{AV} \times \frac{BW}{OW} \times \frac{AD}{BD} = \frac{BR}{AR} \times \frac{AT}{OT} \times \frac{OM}{BM}$$

This equation is true for all triangles.

4.1.4 SOLUTION (2).

As we did before we need to cube the equation 13 to get

$$\left(\frac{h(1-h)}{k(1-k)} \times \frac{p(1-p)}{m(1-m)} \times \frac{t(1-t)}{r(1-r)}\right)^3 = \left(\frac{a}{1-a} \times \frac{1-b}{b} \times \frac{1-v}{v}\right)^3$$

As we did before we also need to cube equation 12 to get

$$\left(\frac{a}{1-a} \times \frac{1-b}{b} \times \frac{1-v}{v}\right)^9 = \left(\left(\frac{m}{1-m}\right) \times \left(\frac{r}{1-r}\right) \times \left(\frac{k}{1-k}\right) \times \left(\frac{1-h}{h}\right) \times \left(\frac{1-t}{t}\right) \times \left(\frac{1-p}{p}\right)\right)^3$$

The next step is then to multiply two equations as shown below

$$((\text{equation 12})^3) \times ((\text{equation 13})^3)$$

On doing this we get

$$\left(\left(\frac{a}{1-a} \times \frac{1-b}{b} \times \frac{1-v}{v} \right)^9 \right) \times \left(\frac{a}{1-a} \times \frac{1-b}{b} \times \frac{1-v}{v} \right)^3 = \left(\frac{1-h}{1-k} \times \frac{1-p}{1-m} \times \frac{1-t}{1-r} \right)^6$$

Which can be simplified to

$$\left(\frac{a}{1-a} \times \frac{1-b}{b} \times \frac{1-v}{v} \right)^2 = \frac{1-h}{1-k} \times \frac{1-p}{1-m} \times \frac{1-t}{1-r}$$

This therefore is our second uncommon solution in this section. The above equation can also be written as shown below

$$\left(\frac{OV}{AV} \times \frac{BW}{OW} \times \frac{AD}{BD} \right)^2 = \frac{WR}{VR} \times \frac{DT}{WT} \times \frac{VM}{DM}$$

This equation is true for all triangles.

4.1.5 SOLUTION (3).

As we did 2 times before we need to cube the equation 13 to get

$$\left(\frac{h(1-h)}{k(1-k)} \times \frac{p(1-p)}{m(1-m)} \times \frac{t(1-t)}{r(1-r)} \right)^3 = \left(\frac{a}{1-a} \times \frac{1-b}{b} \times \frac{1-v}{v} \right)^3$$

For this case we shall not cube equation 12 we shall leave it as is without altering it in any way

$$\left(\left(\frac{a}{1-a} \right) \times \left(\frac{1-b}{b} \right) \times \left(\frac{1-v}{v} \right) \right)^3 = \left(\frac{m}{1-m} \right) \times \left(\frac{r}{1-r} \right) \times \left(\frac{k}{1-k} \right) \times \left(\frac{1-h}{h} \right) \times \left(\frac{1-t}{t} \right) \times \left(\frac{1-p}{p} \right)$$

The next step is then multiply two equations as shown below

$$(\text{equation 12}) \times \frac{1}{(\text{equation 13})^3}$$

On doing this we get

$$\left(\frac{a}{1-a} \times \frac{1-b}{b} \times \frac{1-v}{v} \right)^3 \times \left(\frac{1-a}{a} \times \frac{b}{1-b} \times \frac{v}{1-v} \right)^3 = \left(\frac{(1-k) \times k^2}{(1-h) \times h^2} \times \frac{(1-m) \times m^2}{(1-p) \times p^2} \times \frac{(1-r) \times r^2}{(1-t) \times t^2} \right)^3$$

Which can be simplified to

$$\frac{(1-k) \times k^2}{(1-h) \times h^2} \times \frac{(1-m) \times m^2}{(1-p) \times p^2} \times \frac{(1-r) \times r^2}{(1-t) \times t^2} = 1$$

This therefore is our third uncommon solution in this section. The above equation can also be written as shown below

$$\frac{WR(AR)^2}{VR(BR)^2} \times \frac{DT(OT)^2}{WT(AT)^2} \times \frac{VM(BM)^2}{DM(OM)^2} = 1$$

I am particularly interested with this solution because it shows the relations between the ratios of the cevians with no reference to the ratios of the sides of a triangle.

This equation is true for all triangles.

Solutions 1,2 and 3 describe the diagram below which is a combination of figures 5, 6 &7

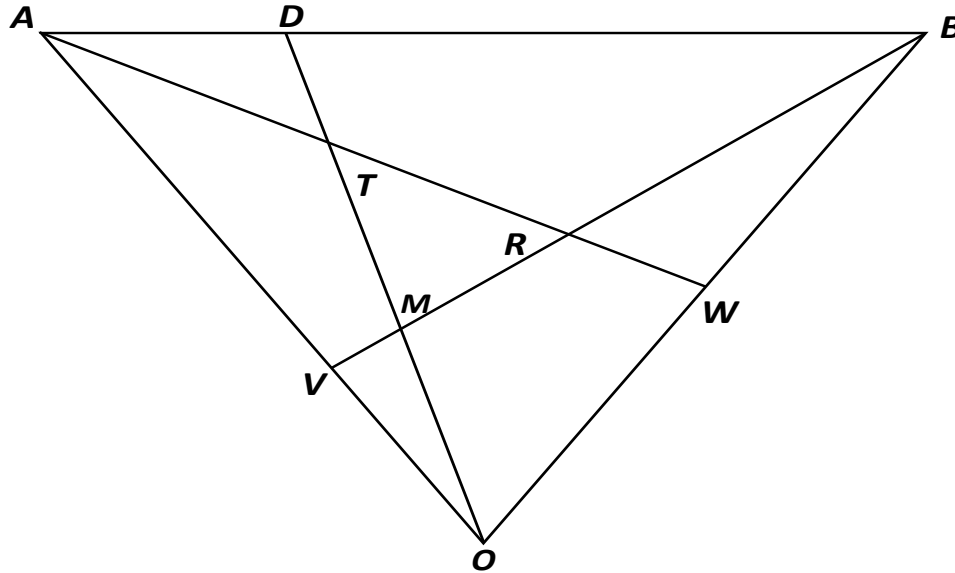


Figure 8: A triangle ABO with 3 non-concurrent cevians: OD, BV &AW. Cevians AW and BV meet at R, cevians AW and OD meet at T and cevians OD and BV meet at M.

In the section below I will attempt to generalize the properties described by the solutions (1, 2&3) in order to come up with generalized identities that are true for all triangles. This however does not make solutions (1, 2&3) useless as they can be used to solve some problems in triangle geometry that involve the parameters stated in the equations.

4.2 GENERALIZATION OF THE PROPERTIES DESCRIBED BY SOLUTIONS (1,2&3) IN RELATION TO CEVIAN TRIANGLES.

In the previous section above we proved 3 identities, that is solutions (1,2&3) that hold true for any triangle (A,B,O). In this last section I will attempt to generalize the properties described by the solutions (1,2&3).

Consider the diagram below.

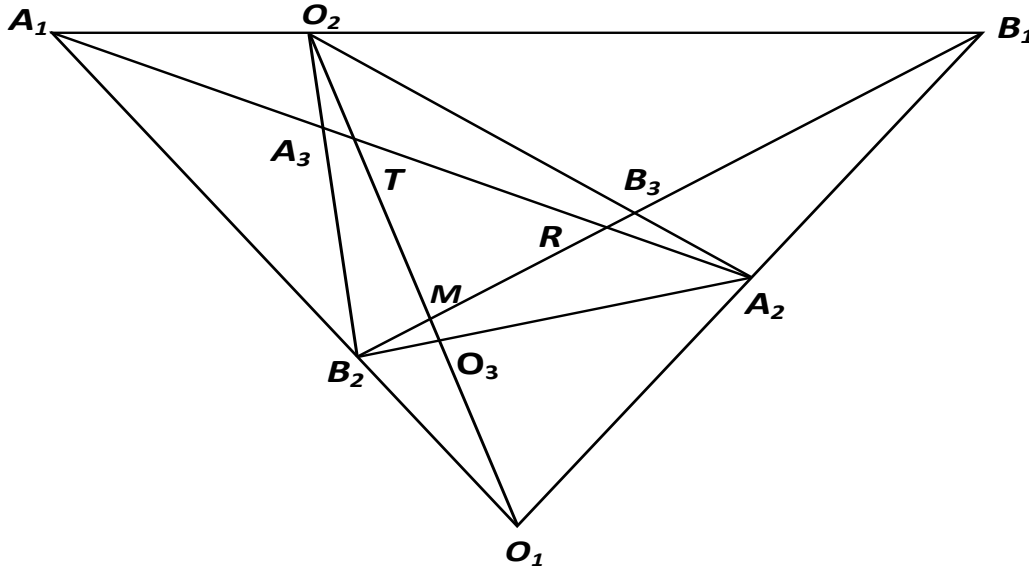


Figure 9: A triangle $O_1B_1A_1$ with 3 non-concurrent cevians: A_1A_2, O_1O_2 & B_1B_2 . The cevians A_1A_2 and O_1O_2 meet at T , the cevians A_1A_2 and B_1B_2 meet at R and the cevians B_1B_2 and O_1O_2 meet at M . The line B_1B_2 meets line O_2A_2 at B_3 and line A_1A_2 meets the line O_2B_2 at A_3 and line O_1O_2 meets line B_2A_2 at O_3 .

It can be shown that the three equations below are true for any such triangle

- I. $\left(\frac{O_1B_2}{A_1B_2} * \frac{B_1A_2}{O_1A_2} * \frac{A_1O_2}{B_1O_2} \right)^2 = \frac{A_3O_2}{B_2A_3} * \frac{B_3A_2}{O_2B_3} * \frac{O_3B_2}{A_2O_3}$, more generally $\left(\frac{O_1B_2}{A_1B_2} * \frac{B_1A_2}{O_1A_2} * \frac{A_1O_2}{B_1O_2} \right)^{2^n} = (b_n)^{(-1)^n}$
- II. $\frac{O_1B_2}{A_1B_2} * \frac{B_1A_2}{O_1A_2} * \frac{A_1O_2}{B_1O_2} = \frac{B_1R}{A_1R} * \frac{A_1T}{O_1T} * \frac{O_1M}{B_1M}$, this equation is in its most general form when considering the triangle (O_1, B_1, A_1) .
- III. $\left(\frac{A_1R}{B_1R} * \frac{O_1T}{A_1T} * \frac{B_1M}{O_1M} \right)^2 = \frac{A_2T}{A_2R} * \frac{O_2M}{O_2T} * \frac{B_2R}{B_2M}$, more generally $\left(\frac{A_1R}{B_1R} * \frac{O_1T}{A_1T} * \frac{B_1M}{O_1M} \right)^{2^n} = a_n$

Indeed by generalizing the properties described by solutions (1, 2&3) from the previous section we should arrive at equations (i, ii & iii) as shown above. In this section we will be proving such equations which serve as identities for any and all triangles. The meaning of the letters (a, b and n) will be explained later on in this section.

4.2.1 PROVING THE GENERALIZED IDENTITIES.

Consider the following diagram.

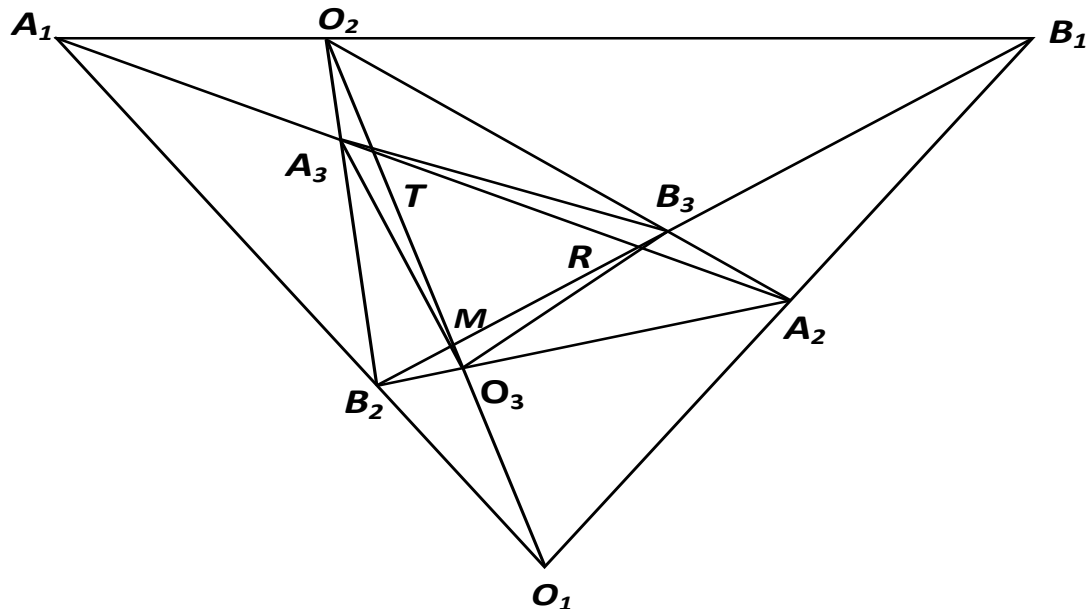


Figure 10: A triangle $O_1B_1A_1$ with 3 non-concurrent cevians: A_1A_2, O_1O_2 & B_1B_2 . The cevians A_1A_2 and O_1O_2 meet at T , the cevians A_1A_2 and B_1B_2 meet at R and the cevians B_1B_2 and O_1O_2 meet at M . The line B_1B_2 meets line O_2A_2 at B_3 and line A_1A_2 meets the line O_2B_2 at A_3 and line O_1O_2 meets line B_2A_2 at O_3 . And the triangle $A_2B_2O_2$ now has another triangle $A_3B_3O_3$ embedded within it.

Indeed our new diagram (O_1, B_1, A_1) is the same to our initial diagram (O, B, A) figure 8 in the previous section.

The only major change is that the naming of the vertices has changed. I have changed the naming of the vertices in order to avoid using many different letters and for simplicity. The other change is the construction of 2 new smaller triangles along the cevians of the original triangle. The essence of these changes will be realized as we proceed with the proofs.

I therefore need to rewrite the solutions (1, 2&3) with respect to triangle (O_1, B_1, A_1) which is our new diagram above. This is shown below.

Solution (3).

$$\left(\frac{A_1R}{B_1R} * \frac{O_1T}{A_1T} * \frac{B_1M}{O_1M} \right)^2 = \frac{A_2T}{A_2R} * \frac{O_2M}{O_2T} * \frac{B_2R}{B_2M}$$

Solution (1).

$$\frac{O_1B_2}{A_1B_2} * \frac{B_1A_2}{O_1A_2} * \frac{A_1O_2}{B_1O_2} = \frac{B_1R}{A_1R} * \frac{A_1T}{O_1T} * \frac{O_1M}{B_1M}$$

Solution (2).

$$\left(\frac{O_1B_2}{A_1B_2} * \frac{B_1A_2}{O_1A_2} * \frac{A_1O_2}{B_1O_2} \right)^2 = \frac{A_2R}{A_2T} * \frac{O_2T}{O_2M} * \frac{B_2M}{B_2R}$$

In our new diagram (O_1, B_1, A_1) along the cevians $(A_1A_2, O_1O_2 \& B_1B_2)$ we can see that more triangles that have the same structure as triangle (O_1, B_1, A_1) can be constructed as shown by triangles (O_2, B_2, A_2) and (O_3, B_3, A_3) . Indeed more triangles can be constructed along the stated cevians indefinitely with an upper limit of triangle (T, M, R) . The points T, R and M can be thought about as our points at infinity in the following manner.

- The point T is a point like O but at infinity.
- The point M is a point like B but at infinity.
- The point R is a point like A but at infinity.

Having constructed such triangles the corresponding diagram is then a series of infinite triangles constructed along the cevians $(A_1A_2, O_1O_2 \& B_1B_2)$. The letter (n) we saw earlier represents the position of a triangle with respect to triangle (O_1, B_1, A_1) in the series of triangles we just discussed, therefore $(n \in \mathbb{Z})$. For example the triangle (O_2, B_2, A_2) is the first triangle after triangle (O_1, B_1, A_1) therefore $(n = 1)$ and the triangle (O_3, B_3, A_3) is the second triangle after triangle (O_1, B_1, A_1) therefore $(n = 2)$. This a crucial fact to consider while constructing our generalized identities.

The next step is to express the relations represented by Solutions (1,2&3) above for triangles (O_2, B_2, A_2) and (O_3, B_3, A_3) , since for Solutions (1,2&3) we were only considering triangles (O_1, B_1, A_1) and (O_2, B_2, A_2) . We shall name these new equations Solutions (4,5&6).

This is shown below.

Solution (4).

$$\left(\frac{O_2M}{O_2T} * \frac{B_2R}{B_2M} * \frac{A_2T}{A_2R}\right)^2 = \frac{A_3R}{A_3T} * \frac{B_3M}{B_3R} * \frac{O_3T}{O_3M}$$

Solution (5).

$$\frac{B_2A_3}{A_3O_2} * \frac{O_2B_3}{B_3A_2} * \frac{A_2O_3}{O_3B_2} = \frac{O_2M}{O_2T} * \frac{B_2R}{B_2M} * \frac{A_2T}{A_2R}$$

Solution (6).

$$\left(\frac{B_2A_3}{A_3O_2} * \frac{O_2B_3}{B_3A_2} * \frac{A_2O_3}{O_3B_2}\right)^2 = \frac{A_3R}{A_3T} * \frac{B_3M}{B_3R} * \frac{O_3T}{O_3M}$$

4.2.2 SOME NECESSARY ADJUSTMENTS TO THE PARAMETERS INVOLVED.

Using solutions (1) to (6) we can come up with the generalized identities we saw earlier but we have to make some necessary adjustments to the equations in order for us to meet this objective. The adjustments we are going to make are substituting some parts of our equations with letters for simplicity and clarity.

This substitution will be done as described below

- Let the letter (a_n) represent all ratio products similar to the ratio product $\left(\frac{A_2T}{A_2R} * \frac{O_2M}{O_2T} * \frac{B_2R}{B_2M}\right)$, which describes a property of the triangles (O_1, B_1, A_1) and (O_2, B_2, A_2) , such that (a_n) represents this same product but with respect to different triangles which are within the series of triangles constructed along the cevians $(A_1A_2, O_1O_2 \& B_1B_2)$, for our case these are (O_2, B_2, A_2) and (O_3, B_3, A_3) , (a_n) would take the form $\left(\frac{A_3R}{B_3R} * \frac{O_3T}{A_3T} * \frac{B_3M}{O_3M}\right)$ where $(n = 2)$. This would then continue for all the triangles constructed along the stated cevians. Note the **numerator is greater than the denominator** for the parameter (a_n) .

- Let the letter (b_n) represent all ratio products similar to the ratio product $\left(\frac{B_2A_3}{A_3O_2} * \frac{O_2B_3}{B_3A_2} * \frac{A_2O_3}{O_3B_2}\right)$, which describes a property of the triangle (O_2, B_2, A_2) , such that (b_n) represents this same product but for different triangles which are within the series of triangles which are constructed along the cevians $(A_1A_2, O_1O_2 \& B_1B_2)$. For example with respect to triangle (O_3, B_3, A_3) , (b_n) would take the form $\left(\frac{B_3A_4}{A_4O_3} * \frac{O_3B_4}{B_4A_3} * \frac{A_3O_4}{O_4B_3}\right)$ where $(n = 2)$. This would then continue for all the triangles constructed along the stated cevians. The **direction** of the products in (b_n) should be the **same** as the one in $\left(\frac{O_1B_2}{A_1B_2} * \frac{B_1A_2}{O_1A_2} * \frac{A_1O_2}{B_1O_2}\right)$.

With the 2 conditions above met the Solutions (1) through (6) can be rewritten as follows:

Solution (3).

$$\left(\frac{A_1R}{B_1R} * \frac{O_1T}{A_1T} * \frac{B_1M}{O_1M}\right)^2 = a_n \text{ where } n = 1$$

Solution (1).

$$\frac{O_1B_2}{A_1B_2} * \frac{B_1A_2}{O_1A_2} * \frac{A_1O_2}{B_1O_2} = \frac{B_1R}{A_1R} * \frac{A_1T}{O_1T} * \frac{O_1M}{B_1M}$$

Solution (2).

$$\left(\frac{O_1B_2}{A_1B_2} * \frac{B_1A_2}{O_1A_2} * \frac{A_1O_2}{B_1O_2}\right)^2 = \frac{1}{a_n} \text{ where } n = 1$$

Solution (4).

$$(a_n)^2 = a_{n+1}$$

Where (n) represents the position of the bigger triangle of the 2 under comparison with respect to the position of triangle (O_1, B_1, A_1) .

Solution (5).

$$(b_n)^{(-1)^{(n+1)}} = a_n$$

Where (n) represents the position of the triangle involved with respect to the position of triangle (O_1, B_1, A_1) .

Solution (6).

$$\left((b_n)^{(-1)^{(n+1)}}\right)^2 = a_{n+1}$$

Where (n) represents the position of the bigger triangle of the 2 under comparison with respect to the position of triangle (O_1, B_1, A_1) .

4.2.3 IDENTITY (I).

Looking closely at parts $\left(\frac{O_1B_2}{A_1B_2} * \frac{B_1A_2}{O_1A_2} * \frac{A_1O_2}{B_1O_2}\right)$ and (b_n) it is obvious that both quantities represent a similar property in their respective triangles. Our task is to find out whether the two quantities are related to each other. From solution (5) we know the equation below is true

$$(b_n)^{-1)^{(n+1)} = a_n \tag{x}$$

From solution (2) we also know that

$$\left(\frac{O_1B_2}{A_1B_2} * \frac{B_1A_2}{O_1A_2} * \frac{A_1O_2}{B_1O_2}\right)^2 = \frac{1}{a_n} \text{ where } n = 1 \tag{y}$$

From equations (x) and (y) we then know the equation below is true

$$\left(\frac{O_1B_2}{A_1B_2} * \frac{B_1A_2}{O_1A_2} * \frac{A_1O_2}{B_1O_2}\right)^2 = \frac{1}{b_n} \text{ where } n = 1$$

Since we are dealing with a series of triangles the generalized identity would thus be

$$\left(\frac{O_1B_2}{A_1B_2} * \frac{B_1A_2}{O_1A_2} * \frac{A_1O_2}{B_1O_2}\right)^{2^n} = (b_n)^{-1)^n$$

It should be noted that this equation is a generalization of some sort of Ceva's theorem since this equation works whether the cevians are concurrent or not. The trigonometric form of this equation is also valid but I will only write it down without its proof, I will leave its derivation as a challenge to the reader.

$$\left(\frac{\sin \angle A_2 O_1 O_3}{\sin \angle B_2 O_1 O_3} * \frac{\sin \angle B_2 A_1 A_3}{\sin \angle O_2 A_1 A_3} * \frac{\sin \angle O_2 B_1 B_3}{\sin \angle A_2 B_1 B_3}\right)^2 = \frac{\sin \angle B_1 B_2 O_2}{\sin \angle B_1 B_2 A_2} * \frac{\sin \angle A_1 A_2 B_2}{\sin \angle A_1 A_2 O_2} * \frac{\sin \angle O_1 O_2 A_2}{\sin \angle O_1 O_2 B_2} \text{ where } n = 1$$

The more general form of the above equation is as shown below.

$$\left(\frac{\sin \angle A_2 O_1 O_3}{\sin \angle B_2 O_1 O_3} * \frac{\sin \angle B_2 A_1 A_3}{\sin \angle O_2 A_1 A_3} * \frac{\sin \angle O_2 B_1 B_3}{\sin \angle A_2 B_1 B_3}\right)^{2^n} = (b_n)^{-1)^n$$

4.2.4 IDENTITY (II).

For triangle (O_1, B_1, A_1) we saw that $\frac{O_1B_2}{A_1B_2} * \frac{B_1A_2}{O_1A_2} * \frac{A_1O_2}{B_1O_2} = \frac{B_1R}{A_1R} * \frac{A_1T}{O_1T} * \frac{O_1M}{B_1M}$

We established the fact that the parts $\frac{O_1B_2}{A_1B_2} * \frac{B_1A_2}{O_1A_2} * \frac{A_1O_2}{B_1O_2}$ and $\frac{B_1R}{A_1R} * \frac{A_1T}{O_1T} * \frac{O_1M}{B_1M}$ should form the parts of our identities that are always constant. The reason for doing this was for the 2 products of ratios stated above to exist only triangle (O_1, B_1, A_1) needed to exist. The others, that is triangles (O_2, B_2, A_2) and (O_3, B_3, A_3) , for this case were somewhat irrelevant. This means that for our above equation the value of (n) does not matter.

For triangle (O_1, B_1, A_1) **identity (ii)** takes the form

$$\frac{O_1B_2}{A_1B_2} * \frac{B_1A_2}{O_1A_2} * \frac{A_1O_2}{B_1O_2} = \frac{B_1R}{A_1R} * \frac{A_1T}{O_1T} * \frac{O_1M}{B_1M}$$

For triangle (O_2, B_2, A_2) **identity (ii)** takes the form

$$\frac{B_2A_3}{A_3O_2} * \frac{O_2B_3}{B_3A_2} * \frac{A_2O_3}{O_3B_2} = \frac{O_2M}{O_2T} * \frac{B_2R}{B_2M} * \frac{A_2T}{A_2R}$$

For triangle (O_3, B_3, A_3) **identity (ii)** takes the form

$$\frac{O_3B_4}{A_3B_4} * \frac{B_3A_4}{O_3A_4} * \frac{A_3O_4}{B_3O_4} = \frac{B_3R}{A_3R} * \frac{A_3T}{O_3T} * \frac{O_3M}{B_3M}$$

Thus we can see from the three equations above that the **identity (ii)** is independent of the value of (n). We can therefore conclude for all triangles in our series of triangles constructed along the cevians $(A_1A_2, O_1O_2 \& B_1B_2)$ the **identity (ii)** triangle is always independent of the value of (n). At this point we can see that the **identity (ii)** is comparing the ratios of the side segments and the ratios of the three cevians in our triangle. It is an obvious fact that in other triangles the ratios involved may not necessarily be arranged in the order I used in my triangle. This thus means for different triangles **identity (ii)** will take different forms as long as the products of the cevians and

line segments involved are correctly matched together. As I did with **identity (i)** above, I will also share the trigonometric version of **identity (ii)** here without its proof.

$$\frac{\sin \angle A_2 O_1 O_3}{\sin \angle B_2 O_1 O_3} * \frac{\sin \angle B_2 A_1 A_3}{\sin \angle O_2 A_1 A_3} * \frac{\sin \angle O_2 B_1 B_3}{\sin \angle A_2 B_1 B_3} = \left(\frac{\sin \angle B_1 A_2 B_3}{\sin \angle O_1 A_2 O_3} * \frac{\sin \angle A_1 O_2 A_3}{\sin \angle B_1 O_2 B_3} * \frac{\sin \angle O_1 B_2 O_3}{\sin \angle A_1 B_2 A_3} \right)^{-1}$$

4.2.5 IDENTITY (III).

Looking closely at parts $\frac{A_1 R}{B_1 R} * \frac{O_1 T}{A_1 T} * \frac{B_1 M}{O_1 M}$ and a_n it is obvious that both quantities represent a similar property in their respective triangles.

From solution (3) we know that

$$\left(\frac{A_1 R}{B_1 R} * \frac{O_1 T}{A_1 T} * \frac{B_1 M}{O_1 M} \right)^2 = a_n \text{ where } n = 1$$

Since we are dealing with a series of triangles that are constructed along the cevians $(A_1 A_2, O_1 O_2 \& B_1 B_2)$, it is then obvious that the generalized identity will take the form shown below.

$$\left(\frac{A_1 R}{B_1 R} * \frac{O_1 T}{A_1 T} * \frac{B_1 M}{O_1 M} \right)^{2^n} = a_n$$

When $(n = 1)$ the **identity (iii)** takes the form.

$$\left(\frac{A_1 R}{B_1 R} * \frac{O_1 T}{A_1 T} * \frac{B_1 M}{O_1 M} \right)^2 = \frac{A_2 T}{A_2 R} * \frac{O_2 M}{O_2 T} * \frac{B_2 R}{B_2 M}$$

The trigonometric version of the equation above takes the form shown below.

$$\left(\frac{\sin \angle B_1 A_2 B_3}{\sin \angle O_1 A_2 O_3} * \frac{\sin \angle A_1 O_2 A_3}{\sin \angle B_1 O_2 B_3} * \frac{\sin \angle O_1 B_2 O_3}{\sin \angle A_1 B_2 A_3} \right)^2 = \frac{\sin \angle O_2 A_3 B_3}{\sin \angle B_2 A_3 O_3} * \frac{\sin \angle A_2 B_3 O_3}{\sin \angle O_2 B_3 A_3} * \frac{\sin \angle B_2 O_3 A_3}{\sin \angle A_2 O_3 B_3} \text{ where } n = 1$$

The general form of the above equation takes the form shown below.

$$\left(\frac{\sin \angle B_1 A_2 B_3}{\sin \angle O_1 A_2 O_3} * \frac{\sin \angle A_1 O_2 A_3}{\sin \angle B_1 O_2 B_3} * \frac{\sin \angle O_1 B_2 O_3}{\sin \angle A_1 B_2 A_3} \right)^{2^n} = a_n$$

With **identities (i, ii & iii)** above I believe I have shown the significance of the 6 linear equations I derived earlier using vectors in exploring some new insights brought up by the same. Indeed I have shown that the 6 linear equations (both the special and generalized versions) we derived can lead to some interesting new insights. The reader should also note that identities **(i, ii & iii)** still hold true when the non-concurrent cevians $(A_1 A_2, O_1 O_2 \& B_1 B_2)$ are found outside the triangle (O_1, B_1, A_1) . I have not proven this in this paper but it can be easily verified using vectors and I thus leave it as a challenge to the reader.

4.2.6 SOME OBSERVATIONS TO NOTE.

In this current investigation I came across other equations in my calculations, that in my view are much simpler compared to the **identities (i, ii &iii)**. However I chose not to use them as the generalized identities because to me they do not contain any known significant ratios in terms of their similarity with other known theorems. For this reason I will only write down these equations without their proofs. The equations below describe figures (9&10).

$$\frac{O_1B_2}{A_1B_2} * \frac{B_1A_2}{O_1A_2} * \frac{A_1O_2}{B_1O_2} = \frac{O_1M}{O_1T} * \frac{A_1T}{A_1R} * \frac{B_1R}{B_1M} = \frac{O_1M}{O_1T} * \frac{O_2T}{O_2M} = \frac{A_1T}{A_1R} * \frac{A_2R}{A_2T} = \frac{B_1R}{B_1M} * \frac{B_2M}{B_2R} \quad (14)$$

From equation (14) this relation is obvious $\frac{O_1M}{O_1T} * \frac{A_1R}{A_1T} = \frac{O_2M}{O_2T} * \frac{A_2R}{A_2T}$

This product of ratios shown above is constant for all triangles constructed along the cevians (A_1A_2, O_1O_2 & B_1B_2) thus the equation can be rewritten in a more general form as shown below.

$$\frac{O_1M}{O_1T} * \frac{A_1R}{A_1T} = l \quad (15)$$

Where (l) is a constant for all the triangles in the triangle series. It should however be noted that that in order for the equation to work the ratios involved must be correctly matched. It should also be noted that the other ratios present in equation (14) can also be used to come up with equations similar to equation (15) the only difference is that the value of (l) will vary depending on the ratios chosen. I believe the equations (14) and (15) can be used to generalize Steiner-Routh's theorem on the relations of the side segments in a triangle and the areas of the triangles.

I believe that similar equations that deal with only side segments exist but in my endeavor I have not been able to prove or derive such ratio-equations.

5.0 CHAPTER 5.

5.1 DISCUSSION.

The results of this study demonstrate the significance of the six equations that were derived by the help of vectors. The equations were used as the basis for proving two commonly known mathematical theorems namely Menelaus' theorem and Ceva's theorem as well as deriving some other three equations that are relatively less common in triangle geometry. I was able to prove these three less common results in triangle geometry that is **identities (i, ii &iii)** with the help of Menelaus' theorem together with the generalized version of the 6 linear equations.

Menelaus' theorem is a theorem about collinear points with reference to a triangle. The theorem shows that if three points on a triangle are collinear, then a certain equation involving ratios of sides and cevians must always be true. In this study, the theorem was first proven using a traditional method and then using the 6 equations derived. The proof using the six equations involved expressing the ratios in terms of the six equations and substituting them into the equation to be proven.

Ceva's theorem is another well-known theorem in triangle geometry that concerns concurrent cevians and the sides of a triangle. The theorem states that if three cevians drawn from the vertices of a triangle are concurrent, then the product of certain ratios involving the sides and the cevians is equal to 1. In this study, Ceva's theorem was also first proven using a traditional method and then using the six equations derived. The proof using the six equations involved expressing the ratios in terms of the six equations and substituting them into the equation to be proved. The resulting equation was then shown to be true, proving Ceva's theorem using the six equations.

In addition to proving these two well-known theorems, a generalized version of the six equations was also used to derive some other equations that are relatively less common in triangle geometry. These equations can be useful in various applications of triangle geometry.

Overall, the results of this study demonstrate the usefulness and significance of the six equations derived including the generalized version of the 6 equations that I wrote down towards the end of this study. By using the equations as a basis for proving well-known theorems and deriving other equations, the study shows how the equations can be applied in various contexts in triangle geometry. The proof of Menelaus' theorem and Ceva's theorem using the six equations also highlights the potential of the equations as a tool for approaching problems in triangle geometry.

Therefore, the equations can be a useful tool for geometric calculations and applications. Here are some potential applications and insights that can be derived from the analysis of the proportions of cevians in a triangle:

Geometry: The analysis of the angle bisectors proportions in a triangle can be applied in various fields of geometry such as trigonometry, calculus, and analytic geometry.

Architecture and Construction: Architects and engineers can use these proportions to design and construct stable and aesthetically pleasing structures. For example, they can use the proportions to determine the size and shape of the base of a structure.

Navigation: The principles of geometry and trigonometry can be used in navigation to calculate distances, directions, and angles. The analysis of angle bisectors proportions can be useful in calculating the angles between two objects, such as two boats in a harbor or two planes in the sky.

Optics: Optics is the study of light and its behavior. The analysis of the proportions of angle bisectors in a triangle can be applied to optics, specifically to the design of lenses and mirrors.

Art: Artists often use the principles of geometry in their artwork. The analysis of angle bisectors proportions in a triangle can be applied in art to create balanced and aesthetically pleasing compositions.

Overall, the analysis of cevian to side proportions in a triangle has a wide range of applications and can provide valuable insights in various fields of study.

5.2 CONCLUSION.

In this research study, I explored the properties of triangles and derived a set of six related linear equations that can be used to prove several mathematical theorems related to triangle geometry. These equations are based on the proportions of the sides and cevians of a triangle and provide a unique and unconventional approach to solving problems in this field.

One of the main contributions of this study is the derivation of two existing mathematical theorems, namely Ceva's theorem and Menelaus' theorem, using the set of six equations we had initially derived. These theorems are fundamental in triangle geometry and provide insights into the relationships between different parts of a triangle. By using our set of 6 equations to prove these theorems, I have demonstrated the usefulness of this approach and shown that it can be applied to other problems in this field as well.

In addition to proving these existing theorems, I have also proven other relatively uncommon results in triangle geometry that can be useful for further research in this area. These results are based on a generalized version of six equations and demonstrate the versatility of this approach. I believe that these findings can contribute to the development of new insights and approaches in triangle geometry and lead to further discoveries in this field.

One of the unique aspects of this approach is the use of vectors to derive the six equations. This approach allowed me to express the equations in a simple and intuitive way and provided a clear and concise method for proving these theorems. I believe that the use of vectors can be useful in other areas of mathematics as well and can provide a powerful tool for solving complex problems.

I believe that the results obtained from this study are significant and can contribute to the understanding of triangle geometry and its applications. This approach provides a fresh perspective on this field and demonstrates the usefulness of unconventional methods for solving mathematical problems.

In conclusion, the set of six related equations I have derived provides a unique and unconventional approach to solving problems in triangle geometry. By using these equations, I have proven two existing mathematical theorems, derived some relatively uncommon results in this field, and demonstrated the usefulness of vectors in this area of study. I believe that my approach can contribute to the development of new insights and approaches in triangle geometry and provide a powerful tool for solving complex problems in this field.

These equations have important applications in various fields, such as geometry, engineering, and physics. For instance, in geometry, they can be used to solve problems related to the geometry of triangles, including finding the location of the in-center and the lengths of angle bisectors. In engineering and physics, they can be used to calculate the forces and torques acting on a structure or an object.

Moreover, the six equations may offer insights into the underlying mathematical relationships among the sides and angles of a triangle, revealing the inherent symmetry and beauty of this fundamental geometric shape. They also demonstrate the power of mathematical proof in revealing more mathematical properties.

At this point it is clear to the reader that in my work I have restricted myself to 2 dimensions. I would like to emphasize that this need not be the case and the idea of involving higher dimensions is indeed valid. Therefore, I encourage further research and exploration of the implications and applications of these 6 equations (both the special and generalized versions), who knows what insights we might get from doing this.

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5.4 DISCLOSURE STATEMENT.

No potential conflict of interest was reported by the author.

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